Faculty of Science, Technology, Engineering and Mathematics
M337 Complex analysis

M337 Solutions to Practice exam 2

There are alternative solutions to many of these questions. Any correct solution that is set out clearly is worth full marks.

Question 1

(a) $\exp(2 + i\pi/4) = e^2 e^{i\pi/4} = \dfrac{e^2}{\sqrt{2}}(1+i)$ 2

(b) $1 + 2\sinh^2(i\pi/6) = \cosh(2i\pi/6) = \cos(\pi/3) = \dfrac{1}{2}$ 2

(c) We have $|-1-i| = \sqrt{2}$ and $\operatorname{Arg}(-1-i) = -3\pi/4$. Hence

$$\operatorname{Log}(-1-i) = \log\sqrt{2} - \frac{3i\pi}{4} = \frac{1}{2}\log 2 - \frac{3i\pi}{4}.$$ 3

(d) We have

$$\operatorname{Log}(-i) = \log 1 - \frac{i\pi}{2} = -\frac{i\pi}{2}.$$

Hence

$$(-i)^{-i} = \exp(-i\operatorname{Log}(-i)) = \exp(-i \times (-i\pi/2)) = e^{-\pi/2}.$$ 3

10 Total

Question 2

(a) (i) Let $f(z) = \dfrac{1}{z^6} = z^{-6}$.

This function is analytic on the region $\mathbb{C} - \{0\}$, which contains the circle $C = \{z : |z| = 1\}$. The function $F(z) = -z^{-5}/5$ is a primitive of f on $\mathbb{C} - \{0\}$. It follows from the Closed Contour Theorem that

$$\int_C \frac{1}{z^6}\,dz = 0.$$ 2

(ii) Let $f(z) = \sin z$.

Then f is analytic on the simply connected region $\mathbb{C}$ and C is a simple-closed contour in $\mathbb{C}$. Since 0 lies inside C, we can apply Cauchy's Integral Formula to give

$$\int_C \frac{\sin z}{z}\,dz = 2\pi i f(0) = 0.$$ 2

(iii) Let $f(z) = \sin z$.

As before, we have that f is analytic on the simply connected region $\mathbb{C}$ and C is a simple-closed contour in $\mathbb{C}$. Since 0 lies inside C, we can apply Cauchy's nth Derivative Formula to give

$$\int_C \frac{\sin z}{z^6}\,dz = \frac{2\pi i}{5!} f^{(5)}(0).$$

Now, $f^{(5)}(z) = \cos z$ and $\cos 0 = 1$. Hence

$$\int_C \frac{\sin z}{z^6}\,dz = \frac{\pi i}{60}.$$ 4

(b) Each of the functions f in part (a) is analytic on $\mathbb{C} - \{0\}$. By the Shrinking Contour Theorem,

$$\int_\Gamma f(z)\,dz = \int_C f(z)\,dz,$$

for any circle Γ centred at the origin. Thus the answers in part (a) remain unchanged if C is replaced by Γ. 2

10 Total

Question 3

(a) The domain of f is $D = \{z : |z| < 1\}$ and the domain of g is $E = \{z : |z| > 1\}$.

The domains of f and g are disjoint, so f and g not direct analytic continuations of each other 1

(b) Observe that f is given by a geometric series in z^2. If $|z| < 1$, then $|z^2| < 1$, so

$$f(z) = \sum_{n=0}^{\infty} (z^2)^n = \frac{1}{1-z^2}.$$

Also, we have

$$g(z) = -\frac{1}{z^2} \sum_{n=0}^{\infty} \left(\frac{1}{z^2}\right)^n,$$

for $|z| > 1$. Since $|z| > 1$ it follows that $|1/z^2| < 1$, so

$$g(z) = -\frac{1}{z^2} \times \frac{1}{1 - 1/z^2} = \frac{1}{1-z^2}.$$

Let

$$h(z) = \frac{1}{1-z^2} \quad (z \in \mathbb{C} - \{-1, 1\}).$$

Then we have shown that $f(z) = h(z)$, for $|z| < 1$, and $g(z) = h(z)$, for $|z| > 1$. 6

(c) We have that

$$(f, D), (h, \mathbb{C} - \{-1, 1\}), (g, E)$$

is a chain of functions, because $D \cap (\mathbb{C} - \{-1, 1\}) = D$ and $E \cap (\mathbb{C} - \{-1, 1\}) = E$, and f and g coincide with h on D and E, respectively, as shown in part (b).

It follows that f and g are analytic continuations of one another. They are not direct analytic continuations, by part (a), so they must be indirect analytic continuations of one another. 3

10 Total

Question 4

(a) Let $f(z) = z^5 + 3z^3 - 1$.

Define $g(z) = z^5$. If $|z| = 2$, then, by the Triangle Inequality,

$$|f(z) - g(z)| = |3z^3 - 1| \le |3z^3| + 1 = 3 \times 2^3 + 1 = 25.$$

Also, for $|z| = 2$, we have $|g(z)| = |z^5| = 32$. Therefore

$$|f(z) - g(z)| < |g(z)|, \quad \text{for } |z| = 2.$$

Since f and g are analytic on the simply connected region $\mathbb{C}$, and $\{z : |z| = 2\}$ is a simple-closed contour in $\mathbb{C}$, we see from Rouché's Theorem that f has the same number of zeros as g inside $\{z : |z| = 2\}$, namely 5.

Now define $g(z) = 3z^3$. If $|z| = 1$, then, by the Triangle Inequality,

$$|f(z) - g(z)| = |z^5 - 1| \le |z^5| + 1 = 2.$$

Also, for $|z| = 1$, we have $|g(z)| = |3z^3| = 3$. Therefore

$$|f(z) - g(z)| < |g(z)|, \quad \text{for } |z| = 1.$$

Since f and g are analytic on the simply connected region $\mathbb{C}$, and $\{z : |z| = 1\}$ is a simple-closed contour in $\mathbb{C}$, we see from Rouché's Theorem that f has the same number of zeros as g inside $\{z : |z| = 1\}$, namely 3.

Furthermore, f has no zeros on the circle $\{z : |z| = 1\}$ since $|f(z) - g(z)| < |g(z)|$ when $|z| = 1$.

It follows that f has $5 - 3 = 2$ zeros inside the annulus $\{z : 1 < |z| < 2\}$. 8

(b) Since f is a polynomial function with real coefficients, it satisfies $\overline{f(z)} = f(\overline{z})$, for all $z \in \mathbb{C}$. Thus any zero z of f in the upper half-plane can be paired with a conjugate zero $\overline{z}$ in the lower half-plane. Now, f has at most five zeros altogether because its degree is five, so it can have at most two zeros in the upper-half plane. 2

Remark In fact, f has exactly two zeros in the upper half-plane, because it has exactly one *real* zero. To see why it has exactly one real zero, consider $f(x) = x^5 + 3x^3 - 1$, for $x \in \mathbb{R}$. This is a real polynomial function of odd degree, so it has at least one real zero (by the Intermediate Value Theorem and the observation that $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$). Next, we have that $f'(x) = 5x^4 + 9x^2$, so $f'(x) > 0$ for $x \ne 0$. It follows that f is an increasing function of x, so it has exactly one real zero.

10 Total

Question 5

(a) Observe that

$$\overline{q}(z) = \frac{1}{z-i}.$$

A complex potential function for the flow is

$$\Omega(z) = \operatorname{Log}(z-i),$$

since this function is a primitive of $\overline{q}$ on $\mathbb{C} - \{x+i : x \le 0\}$. We have

$$\Omega(z) = \log|z-i| + i\operatorname{Arg}(z-i).$$

Hence a stream function for the flow is

$$\Psi(z) = \operatorname{Im}\Omega(z) = \operatorname{Arg}(z-i).$$

The streamlines are given by $\Psi(z) = k$, for real constants k. The streamline through the point $1+2i$, satisfies

$$k = \operatorname{Arg}(1+2i-i) = \operatorname{Arg}(1+i) = \pi/4.$$

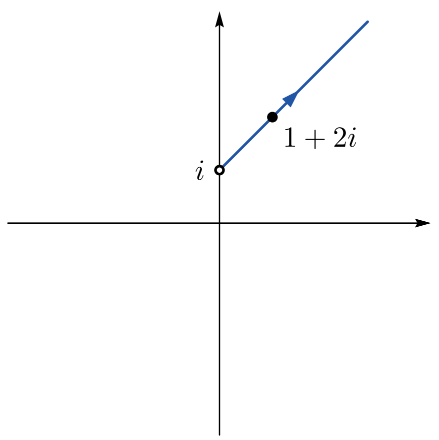
That gives the streamline equation

$$\operatorname{Arg}(z-i) = \frac{\pi}{4}.$$ 5

(b) Since

$$q(1+2i) = \frac{1}{\overline{1+2i}+i} = \frac{1}{1-i} = \frac{1+i}{2},$$

the direction of flow at $1+2i$ is $1+i$.



2

(c) Let Γ be the circle $\{z : |z-i| = 1\}$. The Circulation and Flux Contour Integral tells us that

$$\mathcal{C}_\Gamma + i\mathcal{F}_\Gamma = \int_\Gamma \frac{1}{z-i}\,dz.$$

Applying the Residue Theorem we see that this integral has value $2\pi i \times 1 = 2\pi i$. Hence $\mathcal{C}_\Gamma = 0$ and $\mathcal{F}_\Gamma = 2\pi$, so the point i is a source (of strength 2π). 3

Remark To calculate the circulation and flux, we can choose Γ to be any circle centred at i. The choice does not affect the answer, by the Shrinking Contour Theorem.

10 Total

Question 6

(a) Fixed points of f are solutions z of the equation $f(z) = z$, that is,

$$\frac{1}{2}\left(z + \frac{1}{z}\right) = z,$$

where $z \neq 0$. This equation is equivalent to $z + 1/z = 2z$, or $1/z = z$. Multiplying both sides by z we obtain $z^2 = 1$, which has solutions $z = \pm 1$, the fixed points of f.

Observe that

$$f'(z) = \frac{1}{2}\left(1 - \frac{1}{z^2}\right).$$

Hence $f'(1) = f'(-1) = 0$. Therefore 1 and -1 are both super-attracting fixed points of f. 4

(b) (i) Let $c = -1 - i$. Observe that

$$\begin{aligned} P_c(0) &= -1 - i, \\ P_c^2(0) &= (-1 - i)^2 - 1 - i = -1 + i, \\ P_c^3(0) &= (-1 + i)^2 - 1 - i = -1 - 3i. \end{aligned}$$

So

$$|P_c^3(0)| = \sqrt{1^2 + 3^2} = \sqrt{10} > 2.$$

Hence $-1 - i \notin M$, by HB D2 4.6, p92. 3

(ii) Let $c = -\frac{1}{4}i$. Observe that

$$\left(8\left|-\tfrac{1}{4}i\right|^2 - \tfrac{3}{2}\right)^2 + 8\,\mathrm{Re}\left(-\tfrac{1}{4}i\right) = \left(\tfrac{1}{2} - \tfrac{3}{2}\right)^2 + 0 = 1.$$

Since $1 < 3$, we see from HB D2 4.11(a), p92, that the function P_c has an attracting fixed point. Thus $-\frac{1}{4}i \in M$, by HB D2 4.10, p92. 3

10 Total

Question 7

(a) (i) Let $A = \{z : |z| \leq 1\}$, the closed unit disc. This set is compact because it is closed and bounded. However, $A - \mathbb{R}$ is not compact.

The set $A - \mathbb{R}$ is not compact because it is not closed. It is not closed because the sequence $i/2, i/3, i/4, \ldots$ lies in $A - \mathbb{R}$, but the limit 0 of this sequence does not lie in $A - \mathbb{R}$. 3

(ii) Let $B = \{z : |z| < 1\}$, the open unit disc. This set is a region because it is open and connected. However, $B - \mathbb{R}$ is not a region.

The set $B - \mathbb{R}$ is not a region because it is not connected. It is not connected because there is no path in $B - \mathbb{R}$ joining $i/2$ to $-i/2$. 3

(iii) By definition,

$$C - \mathbb{R} = \{z : z \in C \text{ and } z \notin \mathbb{R}\} = C \cap (\mathbb{C} - \mathbb{R}).$$

The set $\mathbb{C} - \mathbb{R}$ is open, since it is the union of the upper half-plane and the lower half-plane, both basic regions. We are given that C is open, so the intersection $C \cap (\mathbb{C} - \mathbb{R})$ is also open.

Hence $C - \mathbb{R}$ is open. 4

(b) (i) Let $z = x + iy$. Then

$$\begin{aligned} f(z) &= z^2 + 2(\operatorname{Im} z)^2 + 4i(\operatorname{Re} z)^2 \\ &= (x + iy)^2 + 2y^2 + 4ix^2 \\ &= (x^2 + y^2) + i(4x^2 + 2xy). \end{aligned}$$ 1

(ii) Define

$$u(x, y) = x^2 + y^2 \quad \text{and} \quad v(x, y) = 4x^2 + 2xy.$$

Then $f(z) = u(x, y) + iv(x, y)$, and

$$\begin{aligned} \frac{\partial u}{\partial x}(x, y) &= 2x, \\ \frac{\partial u}{\partial y}(x, y) &= 2y, \\ \frac{\partial v}{\partial x}(x, y) &= 8x + 2y, \\ \frac{\partial v}{\partial y}(x, y) &= 2x. \end{aligned}$$

The first Cauchy–Riemann equation is

$$\frac{\partial u}{\partial x}(x, y) = \frac{\partial v}{\partial y}(x, y) \iff 2x = 2x.$$

This equation is satisfied for any complex number z whatsoever.

The second Cauchy–Riemann equation is

$$\begin{aligned} \frac{\partial u}{\partial y}(x, y) = -\frac{\partial v}{\partial x}(x, y) &\iff 2y = -8x - 2y \\ &\iff y = -2x. \end{aligned}$$

Hence both the Cauchy–Riemann equations are satisfied if and only if $y = -2x$, which is the equation of a line through the origin.

Since the partial derivatives exist and are continuous on $\mathbb{C}$, and the Cauchy–Riemann equations are satisfied at all points $z = x + iy$ that satisfy $y = -2x$, we see from the Cauchy–Riemann Converse Theorem that f is differentiable at these points.

Since the Cauchy–Riemann equations are not satisfied at other points, the Cauchy–Riemann Theorem tells us that f is not differentiable at any points of $\mathbb{C} - \{x + iy : y = -2x\}$. 9

20 Total

Question 8

(a) The singularities of f are 0 and 2.

Observe that

$$\begin{aligned}\lim_{z\to 0} zf(z) &= \lim_{z\to 0} \frac{\sin z}{z(z-2)^2} \\ &= \lim_{z\to 0} \frac{\sin z}{z} \times \lim_{z\to 0} \frac{1}{(z-2)^2} \\ &= 1 \times \frac{1}{(-2)^2} = \frac{1}{4}.\end{aligned}$$

Since this limit exists, we see from HB C1 1.2, p59, that f has a simple pole at 0 and

$$\operatorname{Res}(f, 0) = \frac{1}{4}.$$

Next, f has a pole of order 2 at 2. Applying HB C1 1.7, p59, we see that

$$\operatorname{Res}(f, 2) = \lim_{z\to 2} \left(\frac{d}{dz} \left(\frac{\sin z}{z^2} \right) \right).$$

Now

$$\frac{d}{dz} \left(\frac{\sin z}{z^2} \right) = \frac{z^2 \cos z - 2z \sin z}{z^4} = \frac{z \cos z - 2 \sin z}{z^3}.$$

Hence

$$\operatorname{Res}(f, 2) = \frac{2\cos 2 - 2\sin 2}{2^3} = \frac{\cos 2 - \sin 2}{4}.$$ 7

(b) Let $w = z - 1$. Then $z = w + 1$, so

$$g(z) = \frac{3}{z(z-3)} = \frac{3}{(w+1)(w-2)}.$$

Using partial fractions we can write

$$\frac{3}{(w+1)(w-2)} = \frac{A}{w+1} + \frac{B}{w-2},$$

for constants A and B. Multiplying both sides by $(w+1)(w-2)$, we obtain

$$3 = A(w-2) + B(w+1).$$

Setting $w = -1$ gives $3 = -3A$, so $A = -1$. Setting $w = 2$ gives $3 = 3B$, so $B = 1$. Hence

$$\frac{3}{(w+1)(w-2)} = \frac{1}{w-2} - \frac{1}{w+1}.$$

(Check: when $w = 0$, the LHS is $-\frac{3}{2}$ and the RHS is $-\frac{1}{2} - 1 = -\frac{3}{2}$.)

If $1 < |z-1| < 2$, then $1 < |w| < 2$, so

$$g(z) = \frac{1}{w-2} - \frac{1}{w+1} = -\frac{1}{2} \times \frac{1}{1-w/2} - \frac{1}{w} \times \frac{1}{1+1/w},$$

where $|w/2| < 1$ and $|1/w| < 1$.

Hence

$$\begin{aligned} g(z) &= -\frac{1}{2}\left(1 + \frac{w}{2} + \left(\frac{w}{2}\right)^2 + \cdots\right) - \frac{1}{w}\left(1 - \frac{1}{w} + \left(\frac{1}{w}\right)^2 - \cdots\right) \\ &= \left(-\frac{1}{2} - \frac{w}{2^2} - \frac{w^2}{2^3} - \cdots\right) - \left(\frac{1}{w} - \frac{1}{w^2} + \frac{1}{w^3} - \cdots\right) \\ &= \cdots + \frac{1}{w^2} - \frac{1}{w} - \frac{1}{2} - \frac{w}{2^2} - \frac{w^2}{2^3} - \cdots \\ &= \cdots + \frac{1}{(z-1)^2} - \frac{1}{(z-1)} - \frac{1}{2} - \frac{(z-1)}{2^2} - \frac{(z-1)^2}{2^3} - \cdots \\ &= \cdots + \frac{1}{(z-1)^2} - \frac{1}{(z-1)} - \frac{1}{2} - \frac{(z-1)}{4} - \frac{(z-1)^2}{8} - \cdots, \end{aligned}$$

for $1 < |z-1| < 2$. 10

(c) Let $h(z) = f(z) - g(z)$. This is an entire function that is bounded because

$$|h(z)| = |f(z) - g(z)| < 1, \quad \text{for } z \in \mathbb{C}.$$

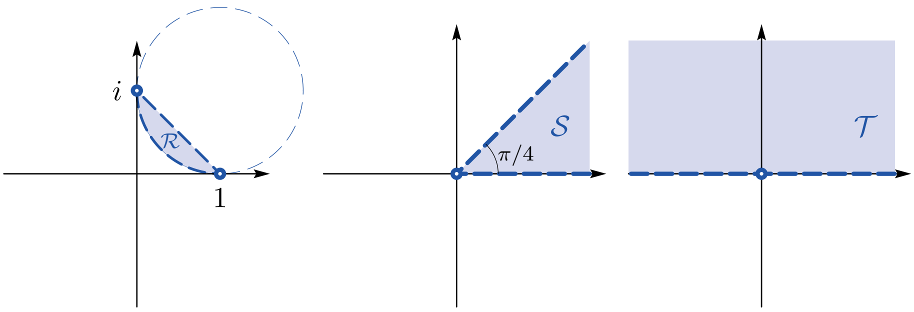
By Liouville's Theorem, h is contant, with value c, say. Then $h(0) = c$ and $h(0) = f(0) - g(0) = 0$, so $c = 0$.

Therefore $f(z) = g(z)$, for $z \in \mathbb{C}$, as required. 3

20 Total

Question 9

(a) 3



(b) Choosing $\lambda = \frac{1}{2}$ gives the point $\frac{1}{2}(1+i)$, which lies on the line segment between 1 and i, which forms one of the boundary arcs of $\mathcal{R}$. 1

(c) Both $\mathcal{R}$ and $\mathcal{S}$ are lunes of angle $\pi/4$. The vertices of $\mathcal{R}$ are 1 and i, and the vertices of $\mathcal{S}$ are 0 and ∞. We can apply the strategy for mapping lunes to find a Möbius transformation f that maps $\mathcal{R}$ onto $\mathcal{S}$.

We choose f such that $f(1) = 0$ and $f(i) = \infty$. Next we choose f to map the point $\frac{1}{2}(1+i)$ on the line segment from 1 to i (with $\mathcal{R}$ to the left) to the point 1 on the half-line from 0 to ∞ (with $\mathcal{S}$ to the left). Since

$$f(1) = 0, \quad f\left(\tfrac{1}{2}(1+i)\right) = 1, \quad f(i) = \infty,$$

we can apply the Explicit Formula for Möbius Transformations to give

$$f(z) = \frac{(z-1)}{(z-i)} \frac{\left(\frac{1}{2}(1+i) - i\right)}{\left(\frac{1}{2}(1+i) - 1\right)} = \frac{(z-1)}{(z-i)} \frac{(1-i)}{(-1+i)}.$$

Hence

$$f(z) = -\frac{z-1}{z-i}.$$

By the strategy, this transformation satisfies $f(\mathcal{R}) = \mathcal{S}$.

Furthermore, because Möbius transformations are one-to-one and conformal on $\widehat{\mathbb{C}}$, we see that f is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$. 7

(d) The function $g(z) = z^4$ quadruples the argument of a complex number and sends the modulus to its fourth power, so it is a one-to-one mapping from the sector $\mathcal{S} = \{z : 0 < \operatorname{Arg} z < \pi/4\}$ onto the sector $\mathcal{T} = \{z : 0 < \operatorname{Arg} z < \pi\}$.

This function g is analytic because it is a polynomial function. Therefore it is a one-to-one conformal mapping from $\mathcal{S}$ onto $\mathcal{T}$. 2

(e) Since f is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$, and g is a one-to-one conformal mapping from $\mathcal{S}$ onto $\mathcal{T}$, the function $h = g \circ f$ is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{T}$. It has rule

$$h(z) = \left(\frac{z-1}{z-i}\right)^4.$$

Next, we have

$$g^{-1}(z) = z^{1/4}.$$

Also,

$$f^{-1}(z) = \frac{-iz-1}{-z-1} = \frac{iz+1}{z+1}.$$

Hence

$$h^{-1}(z) = f^{-1}(g^{-1}(z)) = \frac{iz^{1/4}+1}{z^{1/4}+1}.$$ 5

(f) The function h^{-1} is a one-to-one function because it has an inverse function h.

The function g^{-1} is analytic on $\mathcal{T}$ and the function f^{-1} is analytic on $\mathcal{S}$, so $h^{-1} = f^{-1} \circ g^{-1}$ is analytic on $\mathcal{T}$ by the chain rule.

Thus h^{-1} is a one-to-one analytic mapping from $\mathcal{T}$ to $\mathcal{R}$, so it is a one-to-one conformal mapping from $\mathcal{T}$ onto $\mathcal{R}$ (see HB C3 4.6, p77). 2

20 Total